Energy-consistent, Galerkin approach for the nonlinear dynamics of beams using mixed, intrinsic equations

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The paper presents a Galerkin approach for the solution of the nonlinear beam equations. Nonlinear beam analysis is required when analyzing helicopter blades[1] or high-aspect-ratio wings.[2] The present analysis improves on earlier solution techniques based on nonlinear finite element approach used in Refs. 1 and 2, and is the ideal choice for beam-like structures undergoing large motion. Furthermore, the approach leads to an nonlinear, order-reduction technique which can be applied to the Galerkin equations as well as a finite-element equations.

I. Nonlinear, Intrinsic Beam Equations

The nonlinear, intrinsic, mixed equations for the dynamics of a general (non-uniform, twisted, curved, anisotropic) beam undergoing small strains and large deformation are given below,

$$F' + (\widetilde{\mathbb{k}} + \widetilde{\kappa})F + \mathbb{f} = \dot{P} + \widetilde{\Omega}P \tag{1}$$

$$M' + (\widetilde{\mathbb{k}} + \widetilde{\kappa})M + (\widetilde{\mathbb{e}_1} + \widetilde{\gamma})F + \mathbb{m} = \dot{H} + \widetilde{\Omega}H + \widetilde{V}P \tag{2}$$

$$V' + (\widetilde{\mathbb{k}} + \widetilde{\kappa})V + (\widetilde{\mathbb{e}_1} + \widetilde{\gamma})\Omega = \dot{\gamma} \tag{3}$$

$$\Omega' + (\widetilde{\mathbb{k}} + \widetilde{\kappa})\Omega = \dot{\kappa} \tag{4}$$

where $(\;)'$ denotes the derivative with respect to the undeformed beam reference line and $(\dot{\;})$ denotes the absolute time derivative. $F(x,t)$ and $M(x,t)$ are the measure numbers of the internal force and moment vector (generalized forces), $P(x,t)$ and $H(x,t)$ are the measure numbers of the linear and angular momentum vector (generalized momenta), $\gamma(x,t)$ and $\kappa(x,t)$ are the beam strains and curvatures (generalized strains), $V(x,t)$ and $\Omega(x,t)$ are the linear and angular velocity measures (generalized velocities), and $\mathbb{f}(x,t)$ and $\mathbb{m}(x,t)$ are the external force and moment measures. All measure numbers are calculated in the B–frame, i.e., deformed cross-sectional frame. $\mathbb{k}(x) = \lfloor \mathbb{k}_1(x) \;\; \mathbb{k}_2(x) \;\; \mathbb{k}_3(x) \rfloor$ is the initial twist/curvature of the beam, $\mathbb{e}_1 = \lfloor 1 \;\; 0 \;\; 0 \rfloor^T$.

The first two equations in the above set are the equations of motion[3] while the latter two are the intrinsic kinematical equations[4] derived from the generalized strain-displacement and generalized velocity-displacement equations.

The generalized forces are related to the generalized strains via the cross-sectional beam stiffnesses/flexibilities. These cross-sectional properties can be calculated using an analytical thin-walled theory[5] or computational FEM analysis[6] for general configuration. Such an analysis gives the following, linear constitutive law,

$$\begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} = \begin{bmatrix} \mathbb{R} & \mathbb{S} \\ \mathbb{S}^T & \mathbb{T} \end{bmatrix} \begin{Bmatrix} F \\ M \end{Bmatrix} \tag{5}$$

where, $\mathbb{R}(x)$, $\mathbb{S}(x)$, $\mathbb{T}(x)$, are the cross-sectional flexibilities of the beam. Being linear, the present material law is only valid for small local strains, which can, however, lead to large global deformations.

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The generalized momenta are related to the generalized velocities via the cross-sectional beam inertia,

$$\begin{Bmatrix} P \\ H \end{Bmatrix} = \begin{bmatrix} \mu\Delta & -\mu\widetilde{\xi} \\ \mu\widetilde{\xi} & I \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix} = \begin{bmatrix} \mathbb{G} & \mathbb{K} \\ \mathbb{K}^T & \mathbb{I} \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix} \tag{6}$$

where, $\mu(x)$, $\xi(x)$, $I(x)$ are the mass per unit length, mass center offset (vector in the cross-section from the beam reference axis to the cross-sectional mass center), and mass moment of inertia per unit length respectively.

Usually, the constitutive laws are used to replace some of the variable in terms of other. Here we decided to replace the generalized strains in terms of generalized forces (thus allowing easy specification of zero flexibility) and to replace the generalized momenta by generalized speeds (thus allowing easy specification of zero inertia).

Finally the boundary conditions need to be specified. For the given beam of length $\mathbb{L}$, there will be two boundary conditions at each end:

$$V(\mathbb{0},t) = \mathbb{V}^{\mathbb{0}} \quad \text{OR} \quad F(\mathbb{0},t) = \mathbb{F}^{\mathbb{0}} \tag{7}$$

$$\Omega(\mathbb{0},t) = \mathbb{\Omega}^{\mathbb{0}} \quad \text{OR} \quad M(\mathbb{0},t) = \mathbb{M}^{\mathbb{0}} \tag{8}$$

$$V(\mathbb{L},t) = \mathbb{V}^{\mathbb{L}} \quad \text{OR} \quad F(\mathbb{L},t) = \mathbb{F}^{\mathbb{L}} \tag{9}$$

$$\Omega(\mathbb{L},t) = \mathbb{\Omega}^{\mathbb{L}} \quad \text{OR} \quad M(\mathbb{L},t) = \mathbb{M}^{\mathbb{L}} \tag{10}$$

For the formulation presented we are going to assume a cantilevered beam so as to make it easier to present. It should be noted that the formulation as well as the conclusions that would be presented are general enough to be applicable to all possible BC combinations. Thus, the BC are:

$$V(\mathbb{0},t) = \mathbb{V}^{\mathbb{0}} \tag{11}$$

$$\Omega(\mathbb{0},t) = \mathbb{\Omega}^{\mathbb{0}} \tag{12}$$

$$F(\mathbb{L},t) = \mathbb{F}^{\mathbb{L}} \tag{13}$$

$$M(\mathbb{L},t) = \mathbb{M}^{\mathbb{L}} \tag{14}$$

II. Energy consistent weighting

Now consider the following weighting of all the equations (equations of motion, kinematic equations as well as the boundary conditions). Note that the constitutive equations are not included as these equations are satisfied exactly.

$$\begin{aligned}
\int_{\mathbb{0}}^{\mathbb{L}} \Big\{ & V^T \left[\dot{P} + \widetilde{\Omega} P - F' - (\widetilde{\mathbb{k}} + \widetilde{\kappa}) F - \mathbb{f} \right] \\
& + \Omega^T \left[\dot{H} + \widetilde{\Omega} H + \widetilde{V} P - M' - (\widetilde{\mathbb{k}} + \widetilde{\kappa}) M - (\widetilde{\mathbb{e}_1} + \widetilde{\gamma}) F - \mathbb{m} \right] \\
& + F^T \left[\dot{\gamma} - V' - (\widetilde{\mathbb{k}} + \widetilde{\kappa}) V - (\widetilde{\mathbb{e}_1} + \widetilde{\gamma}) \Omega \right] \\
& + M^T \left[\dot{\kappa} - \Omega' - (\widetilde{\mathbb{k}} + \widetilde{\kappa}) \Omega \right] \Big\} \, dx \\
& - F(\mathbb{0},t)^T \left[V(\mathbb{0},t) - \mathbb{V}^{\mathbb{0}} \right] - M(\mathbb{0},t)^T \left[\Omega(\mathbb{0},t) - \mathbb{\Omega}^{\mathbb{0}} \right] \\
& + V(\mathbb{L},t)^T \left[F(\mathbb{L},t) - \mathbb{F}^{\mathbb{L}} \right] + \Omega(\mathbb{L},t)^T \left[M(\mathbb{L},t) - \mathbb{M}^{\mathbb{L}} \right] = 0
\end{aligned} \tag{15}$$

Integrating by parts and then simplifying the expression we have:

$$\begin{aligned}
\int_{\mathbb{0}}^{\mathbb{L}} & \left[V^T \dot{P} + \Omega^T \dot{H} \right] dx + \int_{\mathbb{0}}^{\mathbb{L}} \left[F^T \dot{\gamma} + M^T \dot{\kappa} \right] dx \\
& = \int_{\mathbb{0}}^{\mathbb{L}} \left[V^T \mathbb{f} + \Omega^T \mathbb{m} \right] dx + \left[V(\mathbb{L},t)^T \mathbb{F}^{\mathbb{L}} + \Omega(\mathbb{L},t)^T \mathbb{M}^{\mathbb{L}} - F(\mathbb{0},t)^T \mathbb{V}^{\mathbb{0}} - M(\mathbb{0},t)^T \mathbb{\Omega}^{\mathbb{0}} \right] = 0
\end{aligned} \tag{16}$$

The first term above is the rate of change of kinetic energy, the second term is the rate of change of potential energy, the third term is the rate of work done (power) due to applied forces in the interior of the beam and the fourth term is the power due to applied forces at the boundaries. The equations states that the rate of change of energy of the beam is equal to the rate of work done on the beam. Thus, the above equations is an energy balance equations. We are going to use the above equation to derive a Galerkin approach to solve the equations.

III. Galerkin approximation

Now let us assume that the primary variables can be written as a linear combination of a set of trial functions.

$$\begin{Bmatrix} V(x,t) \\ \Omega(x,t) \\ F(x,t) \\ M(x,t) \end{Bmatrix} = \begin{bmatrix} V^1(x) & V^2(x) & \cdots & V^n(x) \\ \Omega^1(x) & \Omega^2(x) & \cdots & \Omega^n(x) \\ F^1(x) & F^2(x) & \cdots & F^n(x) \\ M^1(x) & M^2(x) & \cdots & M^n(x) \end{bmatrix} \begin{Bmatrix} q^1(t) \\ q^2(t) \\ \vdots \\ q^n(t) \end{Bmatrix} \tag{17}$$

The above expansion is an approximation to the exact solution and, assuming that the set of trial functions is complete, will approach the exact solution as $n \to \infty$. Now as presented above, the trial functions are coupled, i.e., a given trial function is in general comprised of 12 functions of x for the 12 variables so as to keep the presentation general. One of the first sets of trial functions that is presented next will be a set of independent trial functions for each variable.

The secondary variables (P, H, γ, κ) are linearly related to the primary variables using the constitutive laws, which are exactly satisfied.

$$\begin{Bmatrix} \gamma^i \\ \kappa^i \end{Bmatrix} = \begin{bmatrix} \mathbb{R} & \mathbb{S} \\ \mathbb{S}^T & \mathbb{T} \end{bmatrix} \begin{Bmatrix} F^i \\ M^i \end{Bmatrix} \tag{18}$$

$$\begin{Bmatrix} P^i \\ H^i \end{Bmatrix} = \begin{bmatrix} \mathbb{G} & \mathbb{K} \\ \mathbb{K}^T & \mathbb{I} \end{bmatrix} \begin{Bmatrix} V^i \\ \Omega^i \end{Bmatrix} \tag{19}$$

Now we need n equations which could be used to solve for the n generalized coordinates. These equations come from the above energy balance equation.

$$\begin{aligned} &\int_{\mathbb{0}}^{\mathbb{L}} \Big\{ {V^j}^T \Big[P^i \dot{q}^i + \widetilde{\Omega^k q^k} P^i q^i - {F^i}' q^i - (\widetilde{\mathbb{k}} + \widetilde{\kappa^k q^k}) F^i q^i - \mathbb{f} \Big] \\ &\quad + {\Omega^j}^T \Big[H^i \dot{q}^i + \widetilde{\Omega^k q^k} H^i q^i + \widetilde{V^k q^k} P^i q^i - {M^i}' q^i - (\widetilde{\mathbb{k}} + \widetilde{\kappa^k q^k}) M^i q^i - (\widetilde{\mathbb{e}_1} + \widetilde{\gamma^k q^k}) F^i q^i - \mathbb{m} \Big] \\ &\quad + {F^j}^T \Big[\gamma^i \dot{q}^i - {V^i}' q^i - (\widetilde{\mathbb{k}} + \widetilde{\kappa^k q^k}) V^i q^i - (\widetilde{\mathbb{e}_1} + \widetilde{\gamma^k q^k}) \Omega^i q^i \Big] \\ &\quad + {M^j}^T \Big[\kappa^i \dot{q}^i - {\Omega^i}' q^i - (\widetilde{\mathbb{k}} + \widetilde{\kappa^k q^k}) \Omega^i q^i \Big] \Big\} \, dx \\ &\quad - F^j(\mathbb{0})^T \left[V^i(\mathbb{0}) q^i - \mathbb{V}^{\mathbb{0}} \right] - M^j(\mathbb{0})^T \left[\Omega^i(\mathbb{0}) q^i - \mathbb{\Omega}^{\mathbb{0}} \right] \\ &\quad + V^j(\mathbb{L})^T \left[F^i(\mathbb{L}) q^i - \mathbb{F}^{\mathbb{L}} \right] + \Omega^j(\mathbb{L})^T \left[M^i(\mathbb{L}) q^i - \mathbb{M}^{\mathbb{L}} \right] = 0 \end{aligned} \tag{20}$$

For each $j = 1, 2, \cdots, n$ we have one equations. The energy equation defined in the earlier section is a linear combination of the above equations. Thus if the n equations defined above are satisfied then the energy balance equation is automatically satisfied.

IV. Independent, orthogonal polynomial trial functions

Let all the 12 variables be represented in terms of independent generalized coordinates. Thus:

$$V(x,t) = \begin{bmatrix} \Phi^1(x) & \Phi^2(x) & \cdots & \Phi^\nu(x) \end{bmatrix} \begin{Bmatrix} v^1(t) \\ v^2(t) \\ \vdots \\ v^\nu(t) \end{Bmatrix} \tag{21}$$

$$\Omega(x,t) = \begin{bmatrix} \Phi^1(x) & \Phi^2(x) & \cdots & \Phi^\nu(x) \end{bmatrix} \begin{Bmatrix} \omega^1(t) \\ \omega^2(t) \\ \vdots \\ \omega^\nu(t) \end{Bmatrix} \tag{22}$$

$$F(x,t) = \begin{bmatrix} \Phi^1(x) & \Phi^2(x) & \cdots & \Phi^\nu(x) \end{bmatrix} \begin{Bmatrix} f^1(t) \\ f^2(t) \\ \vdots \\ f^\nu(t) \end{Bmatrix} \tag{23}$$

$$M(x,t) = \begin{bmatrix} \Phi^1(x) & \Phi^2(x) & \cdots & \Phi^\nu(x) \end{bmatrix} \begin{Bmatrix} m^1(t) \\ m^2(t) \\ \vdots \\ m^\nu(t) \end{Bmatrix} \tag{24}$$

where,

$$\Phi^i(x) = \begin{bmatrix} \mathcal{P}^{i-1}(\bar{x}) & 0 & 0 \\ 0 & \mathcal{P}^{i-1}(\bar{x}) & 0 \\ 0 & 0 & \mathcal{P}^{i-1}(\bar{x}) \end{bmatrix} \tag{25}$$

with

$$\bar{x} = \frac{x}{\mathbb{L}} \tag{26}$$

and

$$v^i(t) = \begin{Bmatrix} v_1^i(t) \\ v_2^i(t) \\ v_3^i(t) \end{Bmatrix} \quad \omega^i(t) = \begin{Bmatrix} \omega_1^i(t) \\ \omega_2^i(t) \\ \omega_3^i(t) \end{Bmatrix} \quad f^i(t) = \begin{Bmatrix} f_1^i(t) \\ f_2^i(t) \\ f_3^i(t) \end{Bmatrix} \quad m^i(t) = \begin{Bmatrix} m_1^i(t) \\ m_2^i(t) \\ m_3^i(t) \end{Bmatrix} \tag{27}$$

The independent trial functions used are the shifted Legendre functions.[7] The Legendre functions are a complete set of orthogonal polynomials. The shifted Legendre functions are orthogonal over the shifted (0 - 1) interval as shown below:

$$\int_0^1 \mathcal{P}^i(\bar{x})\mathcal{P}^j(\bar{x})d\bar{x} = \delta_{ij}\frac{1}{2i+1} \tag{28}$$

The polynomials can be iteratively derived as:

$$\mathcal{P}^0(\bar{x}) = 1 \qquad \mathcal{P}^1(\bar{x}) = 2\bar{x} - 1 \tag{29}$$

$$\mathcal{P}^{i+1}(\bar{x}) = \frac{(2i+1)(2\bar{x}-1)\mathcal{P}^i(\bar{x}) - i\mathcal{P}^{i-1}(\bar{x})}{i+1} \tag{30}$$

The Galerkin equations can be derived as:

$$\begin{aligned} \int_{\mathbb{0}}^{\mathbb{L}} \Phi^j \Big[(\mathbb{G}\Phi^i\dot{v}^i + \mathbb{K}\Phi^i\dot{\omega}^i) + \widetilde{\Phi^k\omega^k}(\mathbb{G}\Phi^i v^i + \mathbb{K}\Phi^i\omega^i) - \Phi^{i\prime} f^i - (\widetilde{\mathbb{k}} + \widetilde{\mathbb{S}^T\Phi^k f^k} + \widetilde{\mathbb{T}\Phi^k m^k})\Phi^i f^i - \mathbb{f}\Big]\, dx \\ + \Phi^j(\mathbb{L}) \Big[\Phi^i(\mathbb{L}) f^i - \mathbb{F}^{\mathbb{L}}\Big] = 0 \end{aligned} \tag{31}$$

$$\begin{aligned} \int_{\mathbb{0}}^{\mathbb{L}} \Phi^j \Big[(\mathbb{K}^T\Phi^i\dot{v}^i + \mathbb{I}\Phi^i\dot{\omega}^i) + \widetilde{\Phi^k\omega^k}(\mathbb{K}^T\Phi^i v^i + \mathbb{I}\Phi^i\omega^i) + \widetilde{\Phi^k v^k}(\mathbb{G}\Phi^i v^i + \mathbb{K}\Phi^i\omega^i) - \Phi^{i\prime} m^i \\ -(\widetilde{\mathbb{k}} + \widetilde{\mathbb{S}^T\Phi^k f^k} + \widetilde{\mathbb{T}\Phi^k m^k})\Phi^i m^i - (\widetilde{\mathbb{e}_1} + \widetilde{\mathbb{R}\Phi^k f^k} + \widetilde{\mathbb{S}\Phi^k m^k})\Phi^i f^i - \mathbb{m}\Big]\, dx + \Phi^j(\mathbb{L}) \Big[\Phi^i(\mathbb{L}) m^i - \mathbb{M}^{\mathbb{L}}\Big] = 0 \end{aligned} \tag{32}$$

$$\begin{aligned} \int_{\mathbb{0}}^{\mathbb{L}} \Phi^j \Big[(\mathbb{R}\Phi^k\dot{f}^k + \mathbb{S}\Phi^k\dot{m}^k) - \Phi^{i\prime} v^i - (\widetilde{\mathbb{k}} + \widetilde{\mathbb{S}^T\Phi^k f^k} + \widetilde{\mathbb{T}\Phi^k m^k})\Phi^i v^i - (\widetilde{\mathbb{e}_1} + \widetilde{\mathbb{R}\Phi^k f^k} + \widetilde{\mathbb{S}\Phi^k m^k})\Phi^i\omega^i\Big]\, dx \\ - \Phi^j(\mathbb{0}) \Big[\Phi^i(\mathbb{0}) v^i - V^{\mathbb{0}}\Big] = 0 \end{aligned} \tag{33}$$

$$\int_{\mathbb{0}}^{\mathbb{L}} \Phi^j \Big[(\mathbb{S}^T\Phi^k\dot{f}^k + \mathbb{T}\Phi^k\dot{m}^k) - \Phi^{i\prime}\omega^i - (\widetilde{\mathbb{k}} + \widetilde{\mathbb{S}^T\Phi^k f^k} + \widetilde{\mathbb{T}\Phi^k m^k})\Phi^i\omega^i\Big]\, dx - \Phi^j(\mathbb{0}) \Big[\Phi^i(\mathbb{0})\omega^i - \hat{\Omega}^{\mathbb{0}}\Big] = 0 \tag{34}$$

where, $i, j, k = 1, 2, \cdots, \nu$, and $n = 12 \times \nu$ and the generalized coordinates can be represented as:

$$\begin{Bmatrix} q^1(t) \\ q^2(t) \\ \vdots \\ q^n(t) \end{Bmatrix} = \begin{Bmatrix} v^1(t) \\ \omega^1(t) \\ f^1(t) \\ m^1(t) \\ \vdots \\ \vdots \\ v^\nu(t) \\ \omega^\nu(t) \\ f^\nu(t) \\ m^\nu(t) \end{Bmatrix} \tag{35}$$

V. Special case: constant cross-section, curvature and loading

To obtain the above equations in a form suitable for solution, we need to calculate the integrals. For demonstration, we will assume a constant cross-section ($\mathbb{R}$, $\mathbb{S}$, $\mathbb{T}$, $\mathbb{G}$, $\mathbb{K}$, $\mathbb{I}$ are all constants), constant curvature ($\mathbb{k}$ is constant) and constant distributed loading ($\mathbb{f}$, $\mathbb{m}$ are constants). Furthermore we will assume that the boundary conditions ($\mathbb{F}^{\mathbb{L}}$ or $\mathbb{V}^{\mathbb{L}}$, $\mathbb{M}^{\mathbb{L}}$ or $\Omega^{\mathbb{L}}$, $\mathbb{V}^{\mathbb{0}}$ or $\mathbb{F}^{\mathbb{0}}$, $\Omega^{\mathbb{0}}$ or $\mathbb{M}^{\mathbb{0}}$) are given. With the above assumptions, the equations become:

$$\begin{aligned} \mathcal{A}^{ji}\mathbb{L}(\mathbb{G}\dot{v}^i + \mathbb{K}\dot{\omega}^i) + \mathcal{C}^{jik}\mathbb{L}\widetilde{\omega^k}(\mathbb{G}v^i + \mathbb{K}\omega^i) - \mathcal{B}^{ji}f^i \\ -\mathcal{A}^{ji}\mathbb{L}\tilde{\mathbb{k}}f^i - \mathcal{C}^{jik}\mathbb{L}(\widetilde{\mathbb{S}^T f^k} + \widetilde{\mathbb{T}m^k})f^i - \mathcal{D}^j\mathbb{L}\mathbb{f} + \mathcal{P}^j(1)\mathcal{P}^i(1)f^i - \mathcal{P}^j(1)\mathbb{F}^{\mathbb{L}} = 0 \end{aligned} \tag{36}$$

$$\begin{aligned} \mathcal{A}^{ji}\mathbb{L}(\mathbb{K}^T\dot{v}^i + \mathbb{I}\dot{\omega}^i) + \mathcal{C}^{jik}\mathbb{L}\widetilde{\omega^k}(\mathbb{K}^T v^i + \mathbb{I}\omega^i) + \mathcal{C}^{jik}\mathbb{L}\widetilde{v^k}(\mathbb{G}v^i + \mathbb{K}\omega^i) \\ -\mathcal{B}^{ji}m^i - \mathcal{A}_{ji}\mathbb{L}\tilde{\mathbb{k}}m^i - \mathcal{C}^{jik}\mathbb{L}(\widetilde{\mathbb{S}^T f^k} + \widetilde{\mathbb{T}m^k})m^i - \mathcal{A}^{ji}\mathbb{L}\widetilde{\mathbb{e}_1}f^i \\ -\mathcal{C}^{jik}\mathbb{L}(\widetilde{\mathbb{R}f^k} + \widetilde{\mathbb{S}m^k})f^i - \mathcal{D}^j\mathbb{L}\mathbb{m} + \mathcal{P}^j(1)\mathcal{P}^i(1)m^i - \mathcal{P}^j(1)\mathbb{M}^{\mathbb{L}} = 0 \end{aligned} \tag{37}$$

$$\begin{aligned} \mathcal{A}^{ji}\mathbb{L}(\mathbb{R}\dot{f}^k + \mathbb{S}\dot{m}^k) - \mathcal{B}^{ji}v^i - \mathcal{A}^{ji}\mathbb{L}\tilde{\mathbb{k}}v^i - \mathcal{C}^{jik}\mathbb{L}(\widetilde{\mathbb{S}^T f^k} + \widetilde{\mathbb{T}m^k})v^i \\ -\mathcal{A}^{ji}\mathbb{L}\widetilde{\mathbb{e}_1}\omega^i - \mathcal{C}^{jik}\mathbb{L}(\widetilde{\mathbb{R}f^k} + \widetilde{\mathbb{S}m^k})\omega^i - \mathcal{P}^j(0)\mathcal{P}^i(0)v^i + \mathcal{P}^j(0)\mathbb{V}^{\mathbb{0}} = 0 \end{aligned} \tag{38}$$

$$\begin{aligned} \mathcal{A}^{ji}\mathbb{L}(\mathbb{S}^T\dot{f}^k + \mathbb{T}\dot{m}^k) - \mathcal{B}^{ji}\omega^i - \mathcal{A}^{ji}\mathbb{L}\tilde{\mathbb{k}}\omega^i - \mathcal{C}^{jik}\mathbb{L}(\widetilde{\mathbb{S}^T f^k} + \widetilde{\mathbb{T}m^k})\omega^i dx \\ -\mathcal{P}^j(0)\mathcal{P}^i(0)\omega^i + \mathcal{P}^j(0)\Omega^{\mathbb{0}} = 0 \end{aligned} \tag{39}$$

Summation is implied over indices i and k. And $\mathcal{A}^{ji}$, $\mathcal{B}^{ji}$, $\mathcal{C}^{jik}$ and $\mathcal{D}^j$ are non-dimensional integrals as given below:

$$\mathcal{A}^{ji} = \int_0^1 \mathcal{P}^j(\bar{x})\mathcal{P}^i(\bar{x})d\bar{x} \tag{40}$$

$$\mathcal{B}^{ji} = \int_0^1 \mathcal{P}^j(\bar{x})\left(\mathcal{P}^i(\bar{x})\right)' d\bar{x} \tag{41}$$

$$\mathcal{C}^{jik} = \int_0^1 \mathcal{P}^j(\bar{x})\mathcal{P}^i(\bar{x})\mathcal{P}^k(\bar{x})d\bar{x} \tag{42}$$

$$\mathcal{D}^j = \int_0^1 \mathcal{P}^j(\bar{x})d\bar{x} \tag{43}$$

$$\tag{44}$$

The above set of equations can be written in the following form:

$$A_{ji}\dot{q}_i + B_{ji}q_i + C_{jik}q_iq_k + D_j = 0 \tag{45}$$

VI. Solution

Given the above set of nonlinear equations for the dynamics of beam, we can have the following solutions:

- Nonlinear steady-state solution
- Linear dynamic perturbation solution about the nonlinear steady state
- Nonlinear dynamic solution

A. Nonlinear steady-state solution

The linear steady state solution is calculated by solving:

$$B_{ji}\overline{q}_i + C_{jik}\overline{q}_i\overline{q}_k + \overline{D}_j = 0 \tag{46}$$

where, $\overline{q}$ refers to steady-state solution due to steady-state forcing calculated in $\overline{D}$. The solution is calculated using Newton-Raphson iterations. The Jacobian required for the iteration can be easily calculated as:

$$\mathcal{J}(q) = B_{ji} + C_{jik}q_k + C_{jki}q_k \tag{47}$$

Since the highest order of nonlinearity is second, the Newton-Raphson method leads to converged solution.

B. Nonlinear equations at the steady-state

Now, let the total applied force be given by:

$$D = \overline{D} + \widehat{D}(t) \tag{48}$$

and the corresponding solution be:

$$q = \overline{q} + \widehat{q}(t) \tag{49}$$

Substituting it in the equations of motion we have the equations with respect to the steady-state solution as:

$$\widehat{A}_{ji}\dot{\widehat{q}}_i + \widehat{B}_{ji}\widehat{q}_i + \widehat{C}_{jik}\widehat{q}_i\widehat{q}_k + \widehat{D}_j = 0 \tag{50}$$

where,

$$\widehat{A}_{ji} = A_{ji} \tag{51}$$

$$\widehat{B}_{ji} = B_{ji} + C_{jik}\overline{q}_k + C_{jki}\overline{q}_k \tag{52}$$

$$\widehat{C}_{jik} = C_{jik} \tag{53}$$

The above equation are completely nonlinear with the origin at the steady-state.

C. Linearized equations at the steady-state

The linearized system is given by:

$$\widehat{A}_{ji}\dot{\widehat{q}}_i + \widehat{B}_{ji}\widehat{q}_i + \widehat{D}_j = 0 \tag{54}$$

The linear modes at the steady-state are calculated using the linearized eigenvalue problem at the steady-state:

$$\widehat{A}_{ji}\dot{\widehat{q}}_i + \widehat{B}_{ji}\widehat{q}_i = 0 \tag{55}$$

VII. Modal transformation and reduction

Now consider representing the variables in terms of the modal co-ordinates:

$$\hat{q}_i = U_{il}\xi_l \tag{56}$$

where, $U_{i1},\ U_{i2},\ \cdots$ are the modes of linearized system (Eq. 55). If the length of ξ is less then the length of $\hat{q}$ then we have reduction in the number of states of the system. The reduced order system can be written as:

$$\widehat{A}^r_{ml}\dot{\xi}_l + \widehat{B}^r_{ml}\xi_l + \widehat{C}^r_{mln}\xi_l\xi_n + \widehat{D}^r_m = 0 \tag{57}$$

where,

$$\widehat{A}^r_{ml} = \widehat{A}_{ji}U_{jm}U_{il} \tag{58}$$

$$\widehat{B}^r_{ml} = \widehat{B}_{ji}U_{jm}U_{il} \tag{59}$$

$$\widehat{C}^r_{mln} = \widehat{C}_{jik}U_{jm}U_{il}U_{kn} \tag{60}$$

$$\widehat{D}^r_m = \widehat{D}_j U_{jm} \tag{61}$$

There are two points to be noted. Firstly, the simple form of the modal transformation equation (Eq. 61) is possible due to the specific order of the equations, i.e., the equations are ordered so that the corresponding weighing functions are in the same order as the variables in the generalized coordinate vector.

Secondly, it should be noted that the system equations are in state-space form. Thus, the eigenvalues and eigenvectors of the system will be, in general, complex. Each vibration mode will be represented by two eigenvalues which are complex conjugates and the corresponding eigenvectors which are also complex conjugates. To avoid complex modal coordinates the pair of complex conjugate eigenvectors is represented by two real vectors made up of the real and imaginary parts.[8]

VIII. Results

The equations (full set) were solved for a simple prismatic beam case presented in Table 1. Table 2 lists the calculated frequencies and compares the results with exact results[9] as well as FEM solution.[1] The frequency predictions of a non-rotating as well as rotating beam using the present approach with 10 assumed modes per variable (120 states) are exact to three significant digits. On the other hand, FEM solution with 10 nodes (120 states), is not very accurate leading to errors greater than 10% for the third bending mode.

Fig 1 shows the convergence of the Galerkin approach and compares it to the convergence of the FEM approach. The Galerkin approximation is very accurate and the error decreases sharply with increasing modes. Using around 8 trial functions per variable (less than 100 total variables), we reach the minimum error limit for double precision calculations. The FEM approach has a smooth second-order convergence which means one would require over a million finite elements to get to the error limit.

Fig 2 shows the change in frequency of the beam with applied vertical load. The beam deforms due to the load and the large deformation leads to coupling between the horizontal bending and torsion modes. This is a nonlinear coupling and can be captured by a model with correct nonlinear terms. We retained 4 vertical bending, 2 torsion and 2 horizontal bending modes of the unloaded model in the reduced order model. Thus, the frequencies of the linearized system is exact for no load. As we load the beam the deformation induces nonlinear coupling which is captured by the reduced order model. One could improve on these results by adding perturbation modes[10] or with nonlinear normal modes analysis.[11]

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Table 1. Beam data (for model validation purpose)

Span	16 m
Chord	1 m
Mass per unit length	0.75 kg/m
Mom. Inertia (50% chord)	0.1 kg m
Spanwise elastic axis	50% chord
Center of gravity	50% chord
Bending rigidity	2×10^4 N m^2
Torsional rigidity	1×10^4 N m^2
Bending rigidity (chordwise)	4×10^6 N m^2
Shear/Extensional rigidity	∞

Table 2. Beam structural frequencies (for model validation purpose)

Mode (rad/s)	Exact	Present 10 modes	FEM 10 nodes
Cantilevered Blade: $\omega = 0$ & $v = 0$			
1^{st} bending	2.243	2.243	2.252
2^{nd} bending	14.06	14.06	14.74
3^{rd} bending	39.36	39.36	44.94
1^{st} torsion	31.05	31.05	31.12
2^{nd} torsion	93.14	93.14	95.32
Rotating Cantilevered Blade: $\omega = 3.189\,rad/s$ & $v = 0$			
1^{st} bending	4.114	4.114	4.110
2^{nd} bending	16.23	16.23	16.88
3^{rd} bending	41.59	41.59	47.12
Rotating Cantilevered Blade with Offset: $\omega = 3.189\,rad/s$ & $v = 51.03\,m/s$			
1^{st} bending	5.703	5.703	5.696
2^{nd} bending	18.72	18.72	19.35
3^{rd} bending	44.50	44.50	49.97

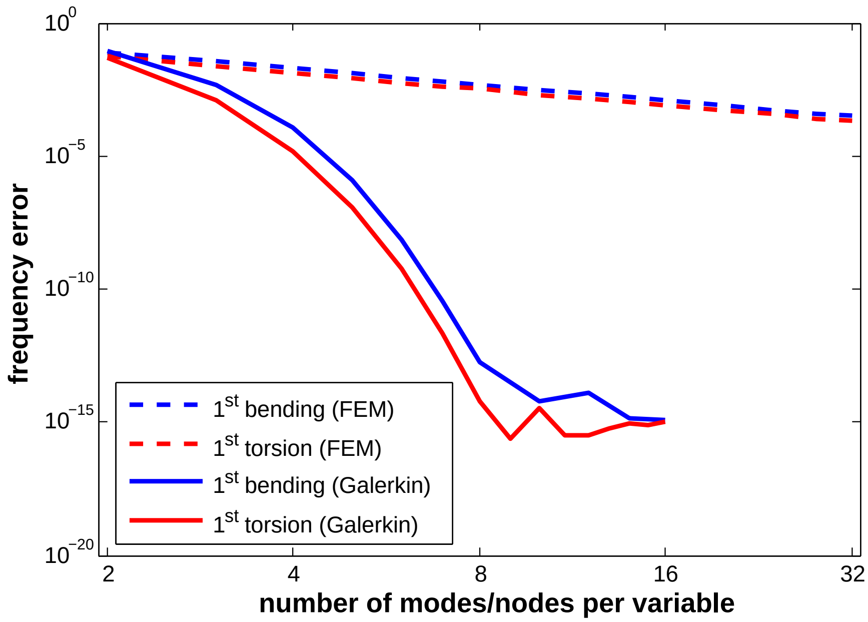

Figure 1. Accuracy of the Galerkin approach

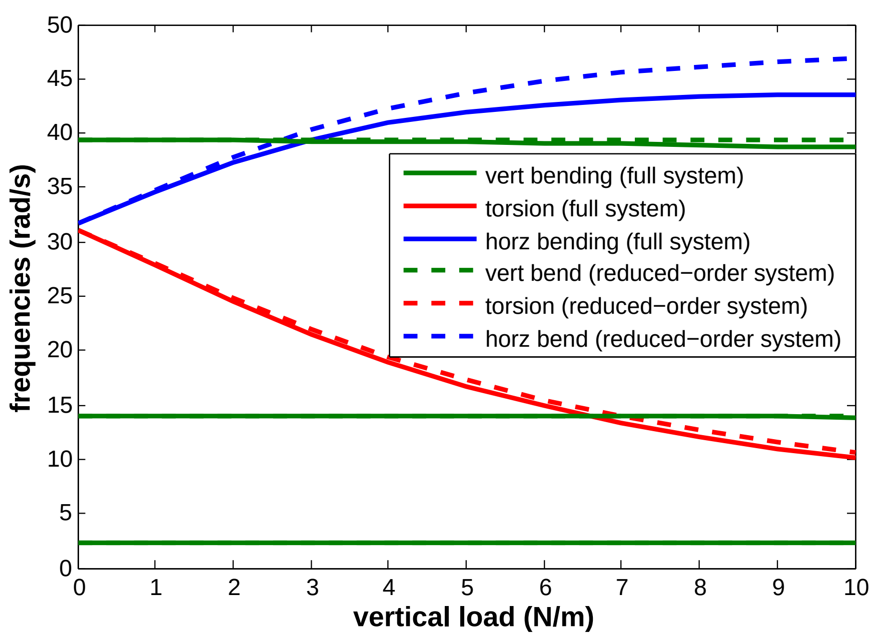

Figure 2. Efficacy of the reduced-order model in capturing nonlinearities